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# INVERTIBILITY OF CONVOLUTION OPERATORS ON HOMOGENEOUS GROUPS

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Abstract. We say that a tempered distribution $A$ belongs to the class $S^m(\mathfrak{g})$ on a homogeneous Lie algebra $\mathfrak{g}$ if its Abelian Fourier transform $a = \widehat{A}$ is a smooth function on the dual $\mathfrak{g}^\star$ and satisfies the estimates

$$|D^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

Let $A \in S^0(\mathfrak{g})$. Then the operator $f \mapsto f \star \widetilde{A}(x)$ is bounded on $L^2(\mathfrak{g})$. Suppose that the operator is invertible and denote by $B$ the convolution kernel of its inverse. We show that $B$ belongs to the class $S^0(\mathfrak{g})$ as well. As a corollary we generalize Melin's theorem on the parametrix construction for Rockland operators.

## Introduction

In a former paper [10] we describe a calculus of a class of convolution operators on a nilpotent homogeneous group $G$ with the Lie algebra $\mathfrak{g}$. These operators are distinguished by conditions imposed on the Abelian Fourier transforms of their kernels similar to those required from the $L^p$-multipliers on $\boldsymbol{R}^n$. More specifically, a tempered distribution $A$ belongs to the class $S^m(G) = S^m(\mathfrak{g})$ if its Fourier transform $a = \widehat{A}$ is a smooth function on the dual to the Lie algebra $\mathfrak{g}^\star$ and satisfies the estimates

$$|D^\alpha a(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}, \qquad \xi \in \mathfrak{g}^\star.$$

In [10] we follow and extend to the setting of a general homogeneous group the ideas of Melin [14] who first introduced such a calculus on the subclass of stratified groups. The classes $S^m(\mathfrak{g})$ of symbols of convolution operators have the expected properties of composition and boundedness (see Propositions 1.3 and 1.4 below) which is a generalization of the results of Melin [14]. However, a complete calculus should also deal with the problem of invertibility. The aim of the present paper is to fill the gap.

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Suppose that $A \in S^0(\mathfrak{g})$. Then, by the boundedness theorem (see Proposition 1.4 below), the operator

$$f \mapsto f \star \widetilde{A}(x) = \int_{\mathfrak{g}} f(xy)A(y)\,dy$$

defined initially on the Schwartz class functions extends uniquely to a bounded operator on $L^2(\mathfrak{g})$. Furthermore, suppose that the operator $f \mapsto f \star \widetilde{A}$ is invertible on $L^2(\mathfrak{g})$ and denote by $B$ the convolution kernel of its inverse. We show here that under these circumstances $B$ belongs to the class $S^0(\mathfrak{g})$ as well. This is done by replacing Melin's techniques of parametrix construction involving the more refined classes $S^{m,s}(\mathfrak{g}) \subset S^m(\mathfrak{g})$ of convolution operators by the calculus of less restrictive classes $S_0^m(\mathfrak{g})$, where no estimates in the central directions are required.

Let us remark that the described result can be also looked upon as a close analogue of the theorem on the inversion of singular integrals, see [9] and Christ-Geller [3].

By using auxiliary convolution operators, namely accretive homogeneous kernels $P^m$ smooth away from the origin, we construct "elliptic" operators $V_1^m$ of order $m > 0$ and get inversion results for classes $S^m(\mathfrak{g})$ for all $m > 0$, which enables us to generalize Melin's theorem on the parametrix construction for Rockland operators. At the same time, however, we present a direct parametrix construction for Rockland operators which avoids the machinery of Melin and also that of the present paper and depends only on well-known properties of Rockland operators as derived in Folland-Stein [7] and the calculus of [10].

We believe that the presented symbolic calculus may be a step towards a more comprehensive pseudodifferential calculus on nilpotent Lie groups parallel to that of Christ-Geller-Głowacki-Polin [4].

## 1. Symbolic calculus.

Let $\mathfrak{g}$ be a nilpotent Lie algebra endowed with a family of dilations $\{\delta_t\}_{t>0}$. We identify $\mathfrak{g}$ with the corresponding nilpotent Lie group by means of the exponential map. Let

$$1 = p_1 < p_2 < \cdots < p_d$$

be the exponents of homogeneity of the dilations. Let

$$\mathfrak{g}_j = \{x \in \mathfrak{g} : tx = t^{p_j} \cdot x\}, \qquad 1 \le j \le d.$$

Denote by $Q = \sum_k \dim \mathfrak{g}_k \cdot p_k$ the homogeneous dimension of $\mathfrak{g}$. Let $|\cdot|$ be a homogeneous norm on $\mathfrak{g}$ and $\rho$ be a smooth function on $\mathfrak{g}$ such that

$$c(1 + |x|) \le \rho(x) \le C(1 + |x|), \qquad x \in \mathfrak{g},$$

for some $C \ge c > 0$. A similar notation will be applied to the dual space $\mathfrak{g}^\star$.

In expressions like $D^\alpha$ or $x^\alpha$ we shall use multiindices

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d),$$

where

$$\alpha_k = (\alpha_{k1}, \alpha_{k1}, \dots, \alpha_{kn_k}),$$

are themselves multiindices with positive integer entries corresponding to the spaces $\mathfrak{g}_k$ or $\mathfrak{g}_k^\star$. The homogeneous length of $\alpha$ is defined by

$$|\alpha| = \sum_{k=1}^{d} |\alpha_k|, \qquad |\alpha_k| = \dim \mathfrak{g}_k \cdot p_k.$$

As usual we denote by $\mathcal{S}(\mathfrak{g})$ or $\mathcal{S}(\mathfrak{g}^\star)$ the Schwartz classes of smooth and rapidly vanishing functions. The Fourier transform

$$\widehat{f}(\xi) = \int_{\mathfrak{g}} f(x)\, e^{-i\langle \xi, x\rangle}\, dx$$

maps $\mathcal{S}(\mathfrak{g})$ onto $\mathcal{S}(\mathfrak{g}^\star)$ and extends to tempered distributions on $\mathfrak{g}$. Let

$$\|f\|^2 = \int_{\mathfrak{g}} |f(x)|^2\, dx, \qquad f \in L^2(\mathfrak{g}).$$

A similar notation will be applied to $f \in L^2(\mathfrak{g}^\star)$, where the Lebesgue measure $d\xi$ on $\mathfrak{g}^\star$ is normalized so that

$$\int_{\mathfrak{g}} |f(x)|^2\, dx = \int_{\mathfrak{g}^\star} |\widehat{f}(x)|^2\, d\xi.$$

The algebra of bounded linear operators on $L^2(\mathfrak{g})$ will be denoted by $\mathcal{B}(L^2(\mathfrak{g}))$.

For a tempered distribution $A$ on $\mathfrak{g}$, we write

$$\mathrm{Op}(A)f(x) = f \star \widetilde{A}(x) = \int_{\mathfrak{g}} f(xy) A(dy), \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Let $m \in \boldsymbol{R}$. By $S^m(\mathfrak{g}) = S^m(\mathfrak{g}, \rho)$ we denote the class of all distributions $A \in \mathcal{S}'(\mathfrak{g})$ whose Fourier transforms $a = \widehat{A}$ are smooth and satisfy the estimates

$$|D^\alpha a(\xi)| \le C_\alpha \rho(\xi)^{m-|\alpha|}, \tag{1.1}$$

where $|\alpha|$ stands for the homogeneous length of a multiindex. Let us recall that $S^m(\mathfrak{g})$ is a Fréchet space with the family of seminorms

$$|a|_\alpha = \sup_{\xi \in \mathfrak{g}^\star} \rho(\xi)^{-m+|\alpha|} |D^\alpha a(\xi)|.$$

It is not hard to see that for every $\varphi \in C_c^\infty(\mathfrak{g})$ equal to 1 in a neighbourhood of 0 the distribution $(1-\varphi)A$ is a Schwartz class function. Thus

$$A = A_1 + F, \tag{1.2}$$

where $A_1$ is compactly supported and $F \in \mathcal{S}(\mathfrak{g})$.

It follows from (1.2) that for every $m \in \boldsymbol{R}$

$$\mathrm{Op}(A) : \mathcal{S}(\mathfrak{g}) \to \mathcal{S}(\mathfrak{g})$$

is a continuous mapping if $A \in S^m(\mathfrak{g})$. Therefore, it extends to a continuous mapping, denoted by the same symbol, of $\mathcal{S}'(\mathfrak{g})$. It is also clear that for $A \in S^m(\mathfrak{g})$ and $B \in S^n(\mathfrak{g})$ the convolution $A \star B$ makes sense and $\mathrm{Op}\,(A \star B) = \mathrm{Op}\,(A)\mathrm{Op}\,(B)$.

The following two propositions have been proved in [10].

**Proposition 1.3.** *If $A \in S^m(\mathfrak{g})$ and $B \in S^n(\mathfrak{g})$, then $A \star B \in S^{m+n}(\mathfrak{g})$ and the mapping*

$$S^m(\mathfrak{g}) \times S^n(\mathfrak{g}) \ni (A, B) \mapsto A \star B \in S^{m+n}(\mathfrak{g})$$

*is continuous.*

**Proposition 1.4.** *If $A \in S^0(\mathfrak{g})$, then $\mathrm{Op}(A)$ is bounded on $L^2(\mathfrak{g})$ and the mapping*

$$S^0(\mathfrak{g}) \ni A \mapsto \mathrm{Op}(A) \in \mathcal{B}(L^2(\mathfrak{g}))$$

*is continuous.*

Let $\mathfrak{z}$ be the central subalgebra corresponding to the largest eigenvalue of the dilations. We may assume that

$$\mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{z}, \qquad \mathfrak{g}^\star = \mathfrak{g}_0^\star \times \mathfrak{z}^\star, \tag{1.5}$$

where $\mathfrak{g}_0$ may be identified with the quotient Lie algebra $\mathfrak{g}/\mathfrak{z}$. The multiplication law in $\mathfrak{g}$ can be expressed by

$$(x, t)(y, s) = (x \circ y, t + s + r(x, y)),$$

where $x \circ y$ is mutiplication in $\mathfrak{g}_0$. Here the variable in $\mathfrak{g}$ has been split in accordance with the given decomposition. In a similar way we also split the variable $\xi = (\eta, \lambda)$ in $\mathfrak{g}^\star$.

Let $m \in \boldsymbol{R}$. By $S_0^m(\mathfrak{g}^\star)$ we denote the class of all distributions $A \in \mathcal{S}'(\mathfrak{g})$ whose Fourier transforms $a = \widehat{A}$ are smooth in the variable $\eta$ and satisfy the estimates

$$|D_\eta^\alpha a(\eta, \lambda)| \leq C_\alpha \rho(\eta, \lambda)^{m-|\alpha|}. \tag{1.6}$$

Again, $S_0^m(\mathfrak{g})$ is a Fréchet space with the family of seminorms

$$|a|_\alpha = \sup_{(\eta,\lambda)\in\mathfrak{g}^\star} \rho(\eta, \lambda)^{-m+|\alpha|} |D_\eta^\alpha a(\eta, \lambda)|.$$

The following result has not been stated explicitly in [10] but follows by the argument given there.

**Proposition 1.7.** *If $A \in S_0^m(\mathfrak{g}^\star)$ and $B \in S_0^n(\mathfrak{g}^\star)$, then $A \star B \in S_0^{m+n}(\mathfrak{g}^\star)$ and the mapping*

$$S_0^m(\mathfrak{g}^\star) \times S_0^n(\mathfrak{g}^\star) \ni (A, B) \mapsto A \star B \in S_0^{m+n}(\mathfrak{g}^\star)$$

*is continuous.*

Let us introduce the following notation:

$$\widehat{f} \# \widehat{g}(\xi) = \widehat{f \star g}(\xi), \qquad \xi \in \mathfrak{g}^\star,$$

for $f, g \in \mathcal{S}(\mathfrak{g})$. Then, for every fixed $\lambda \in \mathfrak{z}^\star$,

$$a \# b(\eta, \lambda) = a(\cdot, \lambda) \#_\lambda b(\cdot, \lambda)(\eta), \tag{1.8}$$

where

$$\widehat{f} \#_\lambda \widehat{g}(\eta) = (f \star_\lambda g)\widehat{\ }(\eta), \qquad f \star_\lambda g(x) = \int_{\mathfrak{g}_0} f(x \circ y^{-1}) g(y)\, e^{i \langle r(x, y^{-1}), \lambda \rangle} dy$$

for $f, g \in \mathcal{S}(\mathfrak{g}_0)$. In particular, $f \star_0 g$ is the usual convolution on the quotient group $\mathfrak{g}_0$.

Let

$$T_{k_j} F(x) = i x_{k_j} F(x), \qquad T_\alpha F(x) = (ix)^\alpha F(x).$$

For a multiindex $\gamma$, let

$$k(\gamma) = \max_{1 \le k \le d} \{k : \gamma_k \ne 0\}, \qquad \gamma \ne 0,$$

and $k(0) = 0$.

**Lemma 1.9.** *Let $f, g \in \mathcal{S}(\mathfrak{g})$. Then for every $\gamma$,*

$$T_\gamma (f \star g) = T_\gamma f \star g + f \star T_\gamma g + \sum_{\alpha, \beta} c^\gamma_{\alpha\beta} T_\alpha f \star T_\beta g \tag{1.10}$$

*or, equivalently, by applying the Fourier transform,*

$$D^\gamma (f \# g) = D^\gamma f \# g + f \# D^\gamma g + \sum_{\alpha, \beta} c^\gamma_{\alpha\beta}\, D^\alpha f \# D^\beta g, \tag{1.11}$$

*where the summation extends over*

$$k(\alpha), k(\beta) \le k(\gamma), \ |\alpha| + |\beta| = |\gamma|, \ \alpha \ne 0, \beta \ne 0. \tag{1.12}$$

*Proof sketch.* This is proved by the following induction. We pick a $\gamma$ and assume that (1.10) holds for all $\gamma' \ne \gamma$ such that $k(\gamma') \le k(\gamma)$ and $|\gamma'| \le |\gamma|$. By (1.22) of Folland-Stein [7], the group law is expressed by

$$(xy^{-1})_k = x_k - y_k + P_k(x, y), \qquad 1 \le k \le d,$$

where the polynomials $P_k$ depend only on variables $x_j, y_j$ with $j < k$. Then

$$T_{x_{k_j}} (f \star g) = \sum_{\alpha, \beta} T_\alpha (f \star T_\beta g), \tag{1.13}$$

where $k(\alpha), k(\beta) < k$ and $|\alpha| + |\beta| = p_k$. We let $T_\gamma = T_{x_{k_j}} T_{\gamma'}$ and apply the induction hypothesis combined with (1.13). Details are left to the reader. $\square$

**Lemma 1.14.** *Let $A \in S^m(\mathfrak{g})$. If $B \in S_0^{-m}(\mathfrak{g})$ is the inverse of $A$, that is,*

$$A \star B = B \star A = \delta_0,$$

*then $B \in S^m(\mathfrak{g})$.*

*Proof.* Let $a = \widehat{A}$, $b = \widehat{B}$. By (1.11),

$$0 = D^{\gamma_d}(a \# b) = D^{\gamma_d} a \# b + a \# D^{\gamma_d} b + \sum c^\gamma_{\alpha\beta} D^\alpha a \# D^\beta b,$$

where the summation extends over $\alpha, \beta$ such that

$$|\alpha| + |\beta| = |\gamma_d|, \ \ |\alpha_d|, |\beta_d| < |\gamma_d|$$

and every multiindex is split as $\alpha = (\alpha', \alpha_d)$, $\alpha_d$ being the part corresponding to $\mathfrak{g}_d^\star$. Therefore,

$$D^{\gamma_d} b = -b \# D^{\gamma_d} a \# b + \sum c^\gamma_{\alpha\beta} b \# D^\alpha a \# D^\beta b,$$

where the sybol on the right-hand side belongs to $\widehat{S}_0^{-m-|\gamma_d|}$ provided that $b \in \widehat{S}_0^{-m-\kappa}$ for $|\kappa| < |\gamma_d|$. By induction, $D^{\gamma_d} b \in \widehat{S}_0^{-m-|\gamma_d|}(\mathfrak{g})$, which is our assertion. $\square$

Let $A_j \in S_0^{m_j}(\mathfrak{g}^\star)$, where $m_j \searrow -\infty$. Then there exists a distribution $A \in S_0^{m_1}(\mathfrak{g}^\star)$ such that

$$A - \sum_{j=1}^N A_j \in S_0^{m_{N+1}}(\mathfrak{g}^\star)$$

for every $N \in \boldsymbol{N}$. The distribution $A$ is unique modulo the class

$$S_0^{-\infty}(\mathfrak{g}^\star) = \bigcap_{n<0} S_0^n(\mathfrak{g}^\star).$$

We shall write

$$A \approx \sum_{j=1}^\infty A_j, \tag{1.15}$$

and call the distribution $A$ the asymptotic sum of the series $\sum A_j$ (cf., e.g., Hörmander [13], Proposition 18.1.3).

We say that $A \in S^m(\mathfrak{g})$, where $m \geq 0$, has a parametrix $B \in S^{-m}(\mathfrak{g})$ if

$$B \star A - \delta_0 \in \mathcal{S}(\mathfrak{g}), \qquad A \star B - \delta_0 \in \mathcal{S}(\mathfrak{g}),$$

where $\delta_0$ stands for the Dirac delta at 0. If $B_1$ is a left-parametrix and $B_2$ a right one, then it is easy to see that $B_1 = B_2$ modulo the Schwartz class functions so both $B_1$ and $B_2$ are parametrices. In particular, if $A$ is symmetric, then either of the conditions implies the other one.

## 2. Sobolev spaces

We say that a tempered distribution $T$ is a *regular kernel of order* $r \in \boldsymbol{R}$, if it is homogeneous of degree $-Q-r$ and smooth away from the origin. A symmetric distribution $T$ is said to be *accretive*, if

$$\langle T, f\rangle \geq 0$$

for real $f \in C_c^\infty(\mathfrak{g})$ which attain their maximal value at 0. Such a $T$ is an infinitesimal generator of a continuous semigroup of subprobability measures $\mu_t$. By the Hunt theory (see, eg., Duflo [5]), $\mathrm{Op}\,(T)$ is a positive selfadjoint operator on $L^2(\mathfrak{g})$ with $\mathcal{S}(\mathfrak{g})$ as its core domain, and for every $0 < m < 1$,

$$\mathrm{Op}\,(T)^m = \mathrm{Op}\,(T^m), \qquad \langle T^m, f\rangle = \frac{1}{\Gamma(-m)} \int_0^\infty t^{-1-m} \langle \delta_0 - \mu_t, f\rangle\, dt,$$

where the distribution $T^m$ is also accretive.

Let $T$ be a fixed symmetric accretive regular kernel of order $0 < m \leq 1$. Then there exists a symmetric nonnegative function $\Omega \in C^\infty(\mathfrak{g} \setminus \{0\})$ which is homogeneous of degree 0 such that

$$\langle T, f\rangle = cf(0) + \lim_{\varepsilon\to} \int_{|x|\geq\varepsilon} \Big(f(0) - f(x)\Big) \frac{\Omega(x)\, dx}{|x|^{Q+m}},$$

where $c \geq 0$. If $c = 0$, $T$ is an infinitesimal generator of a continuous semigroup of probability measures with smooth densities. For every $0 < a < 1$, $T^a$ is also a symmetric regular kernel of order $am$.

Let

$$\langle P, f\rangle = \lim_{\varepsilon\to} \int_{|x|\geq\varepsilon} \frac{f(0) - f(x))\, dx}{|x|^{Q+1}}$$

be a fixed symmetric accretive distribution of order 1. Let us warn the reader that the distributions $P^m$ do not belong to any of the classes $S^m(\mathfrak{g})$ as they do not vanish rapidly at infinity which is a certain technical complication. That is why we introduce the truncated kernels

$$V_0 = I, \qquad V_m = \varphi P^m, \qquad m > 0,$$

where $\varphi$ is a symmetric nonnegative $[0,1]$-valued smooth function with compact support and equal to 1 on the unit ball. Thus defined $V_m \in S^m(\mathfrak{g})$ is also accretive and differs from $P^m$ by a finite measure. Therefore, for every $0 < m \leq 1$, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$(2.1) \quad C_1 \|(I + \mathrm{Op}\,(P))^m f\| \leq \|(I + \mathrm{Op}\,(V_m))f\| \leq C_2 \|(I + \mathrm{Op}\,(P))^m f\|,$$

for $f \in \mathcal{S}(\mathfrak{g})$.

**Proposition 2.2.** *For every* $0 < m \leq 1$*, there exists a constant* $C_m > 0$ *such that*

$$\|f \star V_m\| \geq C_m \|f\|, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

*Proof.* In fact, let $f \in \mathcal{S}(\mathfrak{g})$ and $F = \widetilde{f} \star f$. Then

$$\begin{aligned}\langle f \star V_m, f\rangle &= \langle T, F\rangle \\ &= \lim_{\varepsilon\to} \int_{\varepsilon\le|x|\le 1} \Big(F(0) - \varphi(x)F(x)\Big)\frac{\Omega_m(x)\,dx}{|x|^{Q+1}} + F(0)\int_{|x|\ge 1} \frac{\Omega_m(x)\,dx}{|x|^{Q+1}} \\ &\ge C_m^2 F(0) = C_m^2\|f\|^2\end{aligned}$$

since the first integral is nonnegative. □

It follows from (2.1) and Proposition 2.2 that there exist new constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\|(I + \mathrm{Op}\,(P))^m f\| \le \|\mathrm{Op}\,(V_m)f\| \le C_2\|(I + \mathrm{Op}\,(P))^m f\|, \tag{2.3}$$

for $f \in \mathcal{S}(\mathfrak{g})$ and $0 \le m \le 1$.

Recall from [8] that $P$ is *maximal*, that is, for every regular symmetric kernel $T$ of arbitrary order $m > 0$ there exists a constant $C > 0$ such that

$$\|f \star \widetilde{T}\| \le C\|f \star P^m f\|, \qquad f \in \mathcal{S}(\mathfrak{g}). \tag{2.4}$$

We introduce a scale of Sobolev spaces. For every $m \in \boldsymbol{R}$,

$$H(m) = \{f \in L^2(\mathfrak{g}) : (I + \mathrm{Op}\,(P))^m f \in L^2(\mathfrak{g})\}$$

with the usual norm $\|f\|_{(m)} = \|(I+\mathrm{Op}\,(P))^m f\|_2$. The dual space to $H(m)$ can be identified with $H(-m)$. By (2.3), the norms defined by $V_m$ for $0 < m \le 1$ are equivalent. It follows that for every $0 \le m \le 1$,

$$V_m : H(m) \to H(0)$$

is an isomorphism.

## 3. Main step

Here comes a preliminary version of our theorem.

**Proposition 3.1.** *Let* $0 \le m \le 1$. *Let* $A = A^\star \in S^m(\mathfrak{g})$ *and let* $\mathrm{Op}\,(A) : H(m) \to H(0)$ *be an isomorphism. If* $A \star V_m = V_m \star A$, *then there exists* $B \in S^{-m}(\mathfrak{g})$ *such that*

$$A \star B = B \star A = \delta_0.$$

*In particular* $\mathrm{Op}\,(B) = \mathrm{Op}\,(A)^{-1}$.

By hypothesis, $A$ is invertible in $\mathcal{B}(L^2(\mathfrak{g}))$. There exists a symmetric distribution $B$ such that

$$\mathrm{Op}\,(A)^{-1}f = f \star B, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

We have to show that $B \in S^{-m}(\mathfrak{g})$.

Let $\mathcal{S}_1(\mathfrak{g})$ denote the subspace of $\mathcal{S}(\mathfrak{g})$ consisting of those functions whose Fourier transform is supported where $1 \le |\lambda| \le 2$. Note that this subspace is invariant under convolutions.

**Lemma 3.2.** $\mathrm{Op}\,(B)$ *maps continuously* $\mathcal{S}(\mathfrak{g})$ *into* $\mathcal{S}(\mathfrak{g})$. *The same applies to the invariant space* $\mathcal{S}_1(\mathfrak{g})$.

*Proof sketch.* Being a convolution operator bounded on $L^2(\mathfrak{g})$, $\mathrm{Op}\,(B)$ commutes with right-invariant vector fields $Y$, hence it maps $\mathcal{S}(\mathfrak{g})$ into $L^2(\mathfrak{g}) \cap C^\infty(\mathfrak{g})$. Therefore, it is sufficient to show that for every $\gamma$, $T_\gamma \mathrm{Op}\,(B)$ is bounded in $L^2$-norm. By Lemma 1.9,

$$\begin{aligned} T_\gamma \mathrm{Op}\,(B) &= \mathrm{Op}\,(B)T_\gamma + \mathrm{Op}\,(B)[T_\gamma, \mathrm{Op}\,(A)]\mathrm{Op}\,(B) \\ &= \mathrm{Op}\,(B)T_\gamma + \mathrm{Op}\,(B)\mathrm{Op}\,(A_\gamma)\mathrm{Op}\,(B) \\ &+ \sum_{\alpha,\beta} c_{\alpha\beta} \cdot \mathrm{Op}\,(B)\mathrm{Op}\,(A_\alpha)T_\beta \mathrm{Op}\,(B), \end{aligned} \tag{3.3}$$

where the summation is taken over $\alpha$, $\beta$ as in (1.12). The proof is completed by induction very similar to that of the proof of Lemma 1.9. □

For $n \in \boldsymbol{Z}$, let

$$\langle A_n, f\rangle = 2^{-nm} \int_{\mathfrak{g}} f(2^n x)\, A(dx), \qquad \langle B_n, f\rangle = 2^{nm} \int_{\mathfrak{g}} f(2^n x)\, B(dx).$$

**Corollary 3.4.** *The operators* $\mathrm{Op}\,(B_n)$ *are equicontinuous on* $\mathcal{S}_1(\mathfrak{g})$.

*Proof.* Let $\varphi \in C_c^\infty(\mathfrak{g}_d^\star)$ be equal to 1 for $1 \le |\lambda| \le 2$. Then $\mathrm{Op}\,(A_n)f = \mathrm{Op}\,(A_n')f$, $\mathrm{Op}\,(B_n)f = \mathrm{Op}\,(B_n')f$ for $f \in \mathcal{S}_1(\mathfrak{g})$, where

$$\widehat{A_n'}(\eta,\lambda) = \widehat{A_n}(\eta,\lambda)\varphi(\lambda), \qquad \widehat{B_n'}(\eta,\lambda) = \widehat{B_n}(\eta,\lambda)\varphi(\lambda).$$

By Proposition 1.4, the mapping

$$S^m(\mathfrak{g}) \ni A \to \mathrm{Op}\,(B) \in \mathcal{B}(L^2(\mathfrak{g}))$$

is continuous. Since the family $\{A_n'\}$ is bounded in $S^m(\mathfrak{g})$ so is $\{\mathrm{Op}\,(B_n')\}$ in $\mathcal{B}(L^2(\mathfrak{g}))$. Hence our assertion follows by induction using (3.3). □

Let $a = \widehat{A}$, and let

$$\widehat{A_\lambda}(\eta) = a_\lambda(\eta) = a(\eta,\lambda), \qquad \lambda \in \mathfrak{z}^\star.$$

**Lemma 3.5.** *For every* $f \in \mathcal{S}(\mathfrak{g}_0^\star)$ *the function*

$$\lambda \to \|f \#_\lambda a_\lambda\|^2$$

*is continuous.*

*Proof.* Let $0 < h \in \mathcal{S}(\mathfrak{z}^\star)$ and $h(0) = 1$. Then $F = (f \otimes h)\#a \in \mathcal{S}(\mathfrak{g}^\star)$ and

$$\lambda \to \int_{\mathfrak{g}_0^\star} |F(\eta,\lambda)|^2 d\eta = |h(\lambda)|^2 \|f\#_\lambda a_\lambda\|^2$$

is continuous, which implies our claim. □

From now on we proceed by induction. The assertion of Proposition 3.1 is obviously true in the Abelian case. Let us assume that it holds for $\mathfrak{g}_0 = \mathfrak{g}/\mathfrak{z}$.

**Lemma 3.6.** *The distribution $A_0$ satisfies the hypothesis of Proposition 3.1 on $\mathfrak{g}_0$.*

*Proof.* Observe that under the remaining assumptions of Proposition 3.1 the condition that $\mathrm{Op}\,(A) : H(m) \to H(0)$ is an isomorphism is equivalent to the estimate

$$\|f \star A\| \geq C\|f \star V_m\|, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Now, since $A \star V_m = V_m \star A$, we also have

$$A_0 \star (V_m)_0 = (V_m)_0 \star A_0,$$

where $(V_m)_0$ is the counterpart of $V_m$ on $\mathfrak{g}_0$. Furthermore, we have

$$\|f \star A\| \geq C\|f \star V_m\|$$

so, by Lemma 3.5,

$$\|f_0 \star A_0\| \geq C\|f_0 \star (V_m)_0\|, \qquad f \in \mathcal{S}(\mathfrak{g}),$$

which implies

$$\|f \star A_0\| \geq C\|f \star (V_m)_0\|, \qquad f \in \mathcal{S}(\mathfrak{g}_0).$$

□

Let $b = \widehat{B}$ and $b_n = \widehat{B_n}$. Of course, $b_n \in \mathcal{S}'(\mathfrak{g}^\star)$.

**Lemma 3.7.** *There exist $p \in \widehat{S}_0^{-m}(\mathfrak{g}^\star)$ and $q \in \mathcal{S}(\mathfrak{g}^\star)$ such that*

$$p\#a(\eta,\lambda) = 1 - q(\eta,\lambda), \qquad |\lambda| \leq 2. \tag{3.8}$$

*Proof.* Let $u \in C_c^\infty([0,\infty)$ be equal to 1 in a neighbourhood of $[0,1]$ and supported in $[0,2)$. Then

$$\psi(\eta,\lambda) = u\Big(\frac{\rho(0,\lambda)}{\rho(\eta,0)}\Big)$$

is an element of $\widehat{S}^0(\mathfrak{g}^\star)$. By Lemma 3.6 and the induction hypothesis, there exists $b_0 \in \widehat{S}^{-m}(\mathfrak{g}^\star)$ on $a$ such that

$$b_0 \#_0 a_0 = 1.$$

Let

$$p(\eta,\lambda) = \psi(\eta,\lambda) b_0(\eta).$$

Then $p \in \widehat{S}^{-m}(\mathfrak{g}^\star)$ and

$$\begin{aligned} p\#a(\eta,\lambda) &= p\#(a - a_0)(\eta,\lambda) + b_0\#_0 a_0(\eta) + (1-\psi)(\cdot,\lambda) b_0\#_0 a_0(\eta) \\ &= 1 - q_0(\eta,\lambda), \end{aligned}$$

where for every $\varphi \in C_c^\infty(\mathfrak{z}^\star)$, $\varphi(\lambda)q_0(\eta,\lambda)$ is in $\widehat{S}_0^{-1}(\mathfrak{g}^\star)$. Therefore we take $\varphi \in C_c^\infty(\mathfrak{z}^\star)$ which equals 1 where $|\lambda| \leq 2$ and modify $p_0$ and $q_0$ by letting

$$p_1(\eta,\lambda) = p_0(\eta,\lambda)\varphi(\lambda), \qquad q_1(\eta,\lambda) = q_0(\eta,\lambda)\varphi(\lambda).$$

Now, $p_1 \in \widehat{S}_0^{-m}(\mathfrak{g}^\star)$, $q_1 \in \widehat{S}_0^{-1}(\mathfrak{g}^\star)$, and

$$p_1 \# a = 1 - q_1, \qquad |\lambda| \leq 2.$$

Let

$$p \approx \sum_{k=1}^{\infty} q_1^k \# p_1,$$

where the infinite sum is understood as in (1.15). Then $p \in S_0^{-m}$ and

$$p \# a = 1 - q, \qquad |\lambda| \leq 2,$$

where $q \in \mathcal{S}(\mathfrak{g}^\star)$. ☐

Now we are in a position to conclude the proof of Proposition 3.1. By acting with $b$ on the right on both sides of (3.8), we get

$$b = p + q \# b, \qquad |\lambda| \leq 2,$$

where $q \# b \in \mathcal{S}(\mathfrak{g})$ (Lemma 3.2). Consequently,

$$|D_\eta^\alpha b(\eta, \lambda)| \leq C_\alpha \rho(\eta, \lambda)^{-m-|\alpha|}, \qquad |\lambda| \leq 2.$$

However, the same applies to $b_n$ for every $n \in \boldsymbol{N}$ with the same constants $C_\alpha$. Therefore, by (1.6), $B \in S_0^{-m}(\mathfrak{g})$. Finally, by Lemma 1.14, we conclude that $B \in S^{-m}(\mathfrak{g})$.

**Corollary 3.9.** *Let $A \in S^0(\mathfrak{g})$ and let*

$$\|f \star A\| \geq C\|f\|, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

*There exists $B \in S^0(\mathfrak{g})$ such that*

$$B \star A = \delta_0.$$

*Proof.* In fact,

$$\begin{aligned} C^2\|f\|^2 \leq \|\mathrm{Op}\,(A)f\|^2 &= < \mathrm{Op}\,(A^\star \star A)f, f > \\ &\leq \|\mathrm{Op}\,(A^\star \star A)f\|\|f\|, \end{aligned}$$

for $f \in \mathcal{S}(\mathfrak{g})$ so $\mathrm{Op}\,(A^\star \star A) : L^2(\mathfrak{g}) \to L^2(\mathfrak{g})$ is an isomorphism. By Proposition 3.1 there exists $B_1 \in S^0(\mathfrak{g})$ such that $B_1 \star A^\star \star A = \delta_0$. Therefore $B_1 \star A^\star$ is the left-inverse for $A$. ☐

**Corollary 3.10.** *For every $0 \leq m \leq 1$, there exists $V_{-m} \in S^{-m}(\mathfrak{g})$ such that*

$$V_m \star V_{-m} = V_{-m} \star V_m = \delta_0.$$

## 4. The operator $\mathrm{Op}\,(V_1)$

In this section we show that the role of the family of distributions $V_m \in S^m(\mathfrak{g})$ in defining the Sobolev spaces can be taken over by the family of fractional powers of one single distribution $V_1$. This will enable the final step towards our theorem.

Recall that if a positive selfadjoint operator $A : L^2(\mathfrak{g}) \to L^2(\mathfrak{g})$ is invertible, then

$$A^{-k} f = \frac{\sin k\pi}{\pi} \int_0^\infty t^{-k} (tI + A)^{-1} f \, dt \tag{4.1}$$

for $0 < k < 1$ (see, e.g, Yosida [18], IX.11).

The operator $\mathrm{Op}\,(V_1)$ is positive selfadjoint and invertible. In the proof of the next proposition we follow Beals [2], Theorem 4.9.

**Proposition 4.2.** *For every $m \in \boldsymbol{R}$, $\mathrm{Op}\,(V_1)^m = \mathrm{Op}\,(V_1^m)$, where $V_1^m \in S^m(\mathfrak{g})$.*

*Proof.* It is sufficient to prove the proposition for $-1 < m < 0$. For $t \geq 0$ let

$$R_t = (V_1 + t\delta_0)^{-1}, \qquad r_t = \widehat{R}_t.$$

The operators $\mathrm{Op}\,(V_1) + tI$ satisfy the hypothesis of Proposition 3.1 with the exponent $m = 1$ uniformly so there exist constants $C'_\alpha$ independent of $t$ such that

$$|D^\alpha r_t| \leq C'_\alpha \rho^{-1-|\alpha|}. \tag{4.3}$$

On the other hand

$$tR_t = \delta_0 - R_t \star V_1 \in S^0(\mathfrak{g})$$

uniformly in $t$ so that

$$t|D^\alpha r_t| \leq C''_\alpha \rho^{-\alpha}. \tag{4.4}$$

Combining (4.3) with (4.4) we get

$$|D^\alpha r_t| \leq C_\alpha (t + \rho)^{-1} \rho^{-\alpha}$$

with $C_\alpha$ independent of $t \geq 0$.

Now, the operator $\mathrm{Op}\,(V_1)$ is positive and invertible so, by (4.1), $\mathrm{Op}\,(V_1)^m = \mathrm{Op}\,(V_1^m)$, where

$$(V_1^m)^\wedge = -\frac{\sin m\pi}{\pi} \int_0^\infty t^m r_t \, dt,$$

where $-1 < m < 0$. Therefore

$$\begin{aligned} |D^\alpha (V_1^m)^\wedge| &\leq \frac{C_\alpha}{\pi} \int_0^\infty t^m (t + \rho)^{-1} \, dt \cdot \rho^{-|\alpha|} \\ &\leq C'_\alpha \rho^{m-|\alpha|}, \end{aligned}$$

which proves our case. ☐

**Lemma 4.5.** *Let $K$ be a distribution on $\mathfrak{g}$ smooth away from the origin and satisfying the estimates*

$$|D^\alpha K(x)| \le C_\alpha |x|^{m-Q-|\alpha|}, \qquad x \neq 0, \tag{4.6}$$

*for some $m > 0$. Then,*

$$K = R + \nu,$$

*where $R \in S^{-m}(\mathfrak{g})$ and $\partial\nu \in L^1(\mathfrak{g})$ for every left-invariant differential operator on $\mathfrak{g}$.*

*Proof.* It is sufficient to observe that (4.6) implies that $\widehat{K}$ is smooth away from the origin and

$$|D^\alpha \widehat{K}(\xi)| \le C_\alpha |\xi|^{-m-|\alpha|}, \qquad \xi \neq 0,$$

and let $R = \varphi K$, $\nu = K - R$, where $\varphi \in C_c^\infty(\mathfrak{g})$ is equal to 1 in a neighbourhood of 0. □

Recall that

$$P^m = V_m + \mu,$$

where $V_m \in S^m(\mathfrak{g})$ and $\partial\mu \in L^1(\mathfrak{g})$ for every left-invariant differential operator $\partial$ on $\mathfrak{g}$.

**Proposition 4.7.** *Let $m > 0$. Then*

$$(P^m + \delta_0)^{-1} = R + \nu,$$

*where $R \in S^{-m}(\mathfrak{g})$ and $\partial\nu \in L^1(\mathfrak{g})$ for every left-invariant differential operator $\partial$ on $\mathfrak{g}$.*

*Proof.* Since the kernel $P^m$ is maximal (see (2.4) above), it follows (see Dziubański [6], Theorem 1.13) that the semigroup generated by $P^m$ consists of operators with the convolution kernels

$$h_t(x) = t^{-Q/m} h_1(t^{-1/m}x), \qquad t > 0,$$

which are smooth functions satisfying the estimates

$$|D^\alpha h_t(x)| \le \frac{C_\alpha t}{(t^{1/m} + |x|)^{Q+m+|\alpha|}}, \qquad x \in \mathfrak{g}.$$

Therefore,

$$(P^m + \delta_0)^{-1}(x) = \int_0^\infty e^{-t} h_t(x)\, dt,$$

and consequently satisfies the estimates (1.1). □

We know that there exists a constant $C > 0$ such that

$$C^{-1}\|f \star V_1\| \le \|f \star P\| + \|f\| \le C\|f \star V_1\|,$$

whence

$$\|f \star V_1^m\| \ge C_m \|f\|, \qquad f \in \mathcal{S}(\mathfrak{g}), \tag{4.8}$$

for $m > 0$. Now we have much more.

**Corollary 4.9.** *For every $m > 0$ there exists a constant $C > 0$ such that*

$$C^{-1}\|f \star V_1^m\| \leq \|f \star P^m\| + \|f\| \leq C\|f \star V_1^m\|. \tag{4.10}$$

*Proof.* In fact, we have

$$V_1^m = V_1^m \star (P^m + \delta_0)^{-1} \star (P^m + \delta_0) = \left(V_1^m \star R + V_1^m \star \nu\right) \star (P^m + \delta_0),$$

where $R$ and $\nu$ are as in Proposition 4.7. Then $V_1^m \star R \in S^0(\mathfrak{g})$ and $V_1^m \star \nu \in L^1(\mathfrak{g})$ so

$$\|f \star V_1^m\| \leq C_1(\|f \star P^m\| + \|f\|).$$

The proof of the opposite inequality uses the identity

$$f \star P^m = f \star V_m \star V_1^{-m} \star V_1^m + f \star \mu$$

and (4.8). □

## 5. Main theorem

Here comes our main theorem and the conclusion of its proof.

**Theorem 5.0.1.** *Let $A \in S^m(\mathfrak{g})$, where $m \geq 0$. If $A$ satisfies the estimate*

$$\|f \star A\| \geq C(\|f \star P^m\| + \|f\|), \qquad f \in \mathcal{S}(\mathfrak{g}),$$

*then there exists $B \in S^{-m}(\mathfrak{g})$ such that*

$$B \star A = \delta_0$$

*Conclusion of proof.* Let $A \in S^m(\mathfrak{g})$ satisfy the hypothesis of our theorem. Then $A \star V_1^{-m}$ satisfies the hypothesis of Corollary 3.9 so there exists $B_1 \in S^0(\mathfrak{g})$ such that

$$B_1 \star A \star V_1^{-m} = \delta_0.$$

By acting by convolution with $V_1^m$ on the right and with $V_1^{-m}$ on the left, we see that $B = V_1^{-m} \star B_1$ is the left-inverse for $A$. □

**Corollary 5.1.** *Let $A = A^\star \in S^m(\mathfrak{g})$ for some $m \geq 0$. The following conditions are equivalent:*

1. *There exists $B \in S^{-m}$ such that $B \star A = A \star B = \delta_0$,*
2. *For every $k \in \boldsymbol{R}$,* $\operatorname{Op}(A) : H(k+m) \to H(k)$ *is an isomorphism,*
3. $\operatorname{Op}(A) : H(m) \to H(0)$ *is an isomorphism,*
4. *There exists $C > 0$ such that*

$$\|f \star A\| \geq C(\|f \star P^m\| + \|f\|), \qquad f \in \mathcal{S}(\mathfrak{g}).$$

**Corollary 5.2.** *Let $A \in S^m(\mathfrak{g})$, where $m > 0$, and let* $\operatorname{Op}(A)$ *be positive in $L^2(\mathfrak{g})$. Then $A$ has a parametrix if and only if there exists $C > 0$ such that*

$$||f \star A|| + \|f\| \geq C\|f \star P^m\|, \qquad f \in \mathcal{S}(\mathfrak{g}). \tag{5.3}$$

*Proof.* Let $B \in S^{-m}(\mathfrak{g})$ be a parametrix for $A$. Then

$$B \star A = \delta_0 + h,$$

where $h \in \mathcal{S}(\mathfrak{g})$. Consequently,

$$P^m = V_1^m \star B \star A + g$$

where $g \in L^1(\mathfrak{g})$. Now, $V_1^m \star B \in S^0(\mathfrak{g})$ so it is easy to see that the estimate (5.3) holds.

Suppose now that (5.3) holds true. Since $A$ is positive this implies

$$\|f \star P^m\| \leq C_1 \|f \star (A + \delta_0)\|,$$

which, by Corollary 5.1, implies that $A + \delta_0 \in S^m(\mathfrak{g})$ has an inverse $B_1 \in S^{-m}$. Thus

$$B_1 \star A = \delta_0 - B_1,$$

and the parametrix $B$ can be found as an asymptotic series

$$B \approx \sum_{k=1}^{\infty} B_1^k.$$

□

## 6. Rockland operators

A left-invariant homogeneous differential operator $R$ is said to be a *Rockland operator* if for every nontrivial irreducible unitary representation $\pi$ of $\mathfrak{g}$, $\pi_R$ is injective on the space of $C^\infty$-vectors of $\pi$.

Let $R$ be a left-invariant differential operator homogeneous of degree $\lambda = -Q - m$, that is,

$$R(f \circ \delta_t) = t^m Rf, \qquad f \in \mathcal{S}(\mathfrak{g}), \quad t > 0.$$

It is well-known that the following conditions are equivalent:

(1) $R$ is a Rockland operator,
(2) $R$ is hypoelliptic,
(3) For every regular kernel $T$ of order $m$, there exists a constant $C > 0$ such that

$$\|\mathrm{Op}\,(T)f\| \leq C\|Rf\|, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

That (1) is equivalent to (2) was proved by Helffer-Nourrigat [12] with a contribution from Beals [1] and Rockland [16]. Helffer-Nourrigat [12] also contains the proof of equivalence of (1)-(3) for $\mathrm{Op}\,(T)$ being a differential operator. The remaining part was obtained by the author in [8] and [11].

It has been proved by Melin [14] that a Rockland operator on a *stratified* homogenenous group has a parametrix. We are going to show that in fact this is so on any homogeneous group.

**Corollary 6.1.** *A Rockland operator on $\mathfrak{g}$ has a parametrix.*

*Proof.* Without any loss of generality we may assume that $R$ is positive. Then the assertion follows from (3) and Corollary 5.2. ☐

Thus we have one more condition equivalent to (1)-(3). However, the techniques of the present paper can be applied directly to Rockland operators rending unnecessary any reference to Theorem 5.0.1 or Corollary 5.2. What is needed are well-known properties of Rockland operators and the symbolic calculus of Proposition 1.3. Here is a brief sketch of a direct parametrix construction for a Rockland operator $R$.

We may assume that $R$ is positive. By Folland-Stein [7], Chapter 4.B, $R$ is essentially selfadjoint on $L^2(\mathfrak{g})$ with $\mathcal{S}(\mathfrak{g})$ for its core domain. Moreover, the semigroup generated by it consists of convolution operators with kernels

$$p_t(x) = t^{-Q/m} p_1(t^{-1/m}x),$$

where $p_1$ is a Schwartz class function. Note that $R = \mathrm{Op}\,(R\delta_0)$. Let $S = (\delta_0 + R\delta_0)^{-1}$. It follows that

$$\widehat{S}(\xi) = \int_0^\infty e^{-t}\widehat{p_1}(t^{1/m}\xi)\,dt$$

is a smooth function satisfying the estimates which show that $S \in S^{-m}(\mathfrak{g})$. Moreover,

$$S \star R\delta_0 = \delta_0 - S,$$

and by the usual argument the asymptotic series

$$S_1 \approx \sum_{k=1}^{\infty} S^k$$

defines a parametrix for $R$ (cf. Melin [14]).

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